Thanks for looking at our Optimization problem examples

To the Right

Table of the types of optimization problems solved in Chapter 5.

Types of Optimization Problems Solved in Chapter 5

Optimization With One Variable
- Example 5.1 - Play Yard
- Example 5.2 - Maximize $a^2b^2$
- Example 5.3 - Maximize Profits
- Example 5.4 - Dog Kennels
- Example 5.5 - Cylindrical Tank
- Example 5.6 - Right Triangle
- Example 5.7 - Rectangle
- Example 5.8 - Shortest Time

Optimization With Two Variables
- Example 5.9 - Two Symbolic Variables
- Example 5.10 - Rectangular Tank (Cost)

Linear Programming
- Example 5.10 - Solution at the Corners
- Example 5.11 - Rectangular Tank

Below

Three sample problems are presented. They have three levels of difficulty

• Easier
• Moderate
• Harder

Scroll or page down for the sample.s

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**Problem:** A child’s rectangular play yard is to be built next to the house. To make the three sides of the playpen, twenty-four feet of fencing are available. What should be the dimensions of the sides to make a maximum area?

**Solution:**

**Preparation:** Know the problem thoroughly

- Read the problem statement again, carefully.

A child’s rectangular play yard is to be built next to the house. To make the three sides of the playpen, twenty-four feet of fencing are available. What should be the dimensions of the sides to make a maximum area?

- Restate the given information clearly

  3 sided play yard - rectangular
  24 feet of fencing available
  Make area as large as possible.

- In words, write what is to be found

  Dimensions of pen for maximum area

**Translation** into an optimization formulation.

- Drawing (As important as ever!)

  ![House and Play Yard Diagram]

- Define Symbols: (Also as important as ever!)

  We can add some symbols (and units) to the drawing.

  ![Symbol Diagram]

- Other Symbols:

  \[ A = \text{total area of yard} = x \times y \text{ (sq. feet)} \]
  \[ L = \text{total length of fence} \quad \text{(feet)} = 24 \]

**Application:** Solution by Graphing

Construct a graph of the criterion function, A, and the (now) single design variable, x. A table of values is:

<table>
<thead>
<tr>
<th>x (ft)</th>
<th>y = 24 - 2x (ft)</th>
<th>ft²</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>64</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>72</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>64</td>
</tr>
</tbody>
</table>

Note: x and y are in feet. A is in ft².
We can see from just this much that the maximum is between x = 4 and x = 8, so we try some more values between 4 and 8:

<table>
<thead>
<tr>
<th>x</th>
<th>y = 24 - 2x</th>
<th>A = xy</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>16</td>
<td>64</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>70</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>72</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>70</td>
</tr>
</tbody>
</table>

Let’s make the graph. (Students make a nice, big one.)

The symmetry around x = 6 leads us to suspect that the optimum is at x = 6, where A = 72 ft². We could verify that assumption with trials at x = 5.9 and 6.1. The maximum is at x = 6 and y = 12 ft.
**Problem:** An open-top cylindrical tank with a volume of ten cubic feet is to be made from a sheet of steel. Find the dimensions of the tank that will require as little material used in the tank as possible.

**Solution:**

1 - **Preparation** Re-reading the problem statement and writing down what is given and what is to be found.

"An open-top cylindrical tank with a volume of ten cubic feet is to be made from a sheet of steel. Find the dimensions of the tank that will require as little material used in the tank as possible."

- Rewrite the given info
  - A cylindrical tank open at the top.
  - Made from as little sheet material as possible.
  - Volume is 10 ft$^3$.

- To Find Dimension of the tank that minimizes the sheet material needed.

2 - **Translation** to an optimization problem:

"As little material as possible" translates to "as little outside surface area of the tank as possible." Or better: "minimum outside surface area."

- **Drawings:**
  - Symbols: Four definitions are in the drawing above.
    - Volume = $V$ ft$^3$.
    - Surface Area = $A$ ft$^2$
    - $r =$ radius of tank (ft)
    - $h =$ height of tank (ft)

**Formulation as an optimization problem**

Design Variables: $r$, radius (ft) $h$, height (ft)

The Criterion Function in this example is the surface area, $A$, since when it is minimum, the amount of material used will be a minimum.

Criterion Function $= A = \pi r^2 + 2 \pi r h$

A Constraint in this case is the required volume.

Constraint: $V = \pi r^2 \ h = 10 \ ft^3$.

Or: $h = 10 / \pi r^2$

Now we have the optimization problem in mathematical form. In words, the math problem is the find the values of $r$ and $h$ that give a tank of minimum surface area with a volume of 10 ft$^3$.

3 - **Application:** Solution by Numerical Search.

We can almost always proceed by systematic numerical search.

In this example, there is really only a single unknown design variable because the constraint on the volume can be used. The table below shows a solution found by trying a value for $r$, computing $h$ from the volume, and then computing the resulting surface area, $A$. This table was produced by using a spreadsheet program which is most convenient for such problems, but not really necessary in such a simple one-variable search.

One can start anywhere, but starting with $r=1$ seems a reasonable place. After the third trial, it is apparent that the minimum $A$ is between $r=1$ and $r=3$.

Thus we plan to try $r=1.5$ and $r=2.5$. However, the result of $r=1.5$ reveals that the minimum $A$ will be between $r=1$ and $r=2$. We continue with this process as shown.

<table>
<thead>
<tr>
<th>Trial</th>
<th>$r$</th>
<th>$h = 10/\pi r^2$</th>
<th>$A = \pi r^2 + 20/r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3.18</td>
<td>23.14</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.796</td>
<td>22.56</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.35</td>
<td>34.93</td>
</tr>
<tr>
<td>4</td>
<td>1.5</td>
<td>1.414</td>
<td>20.40</td>
</tr>
<tr>
<td>5</td>
<td>1.25</td>
<td>2.04</td>
<td>20.91</td>
</tr>
<tr>
<td>6</td>
<td>1.75</td>
<td>1.04</td>
<td>21.04</td>
</tr>
<tr>
<td>7</td>
<td>1.4</td>
<td>1.62</td>
<td>20.44</td>
</tr>
<tr>
<td>8</td>
<td>1.3</td>
<td>1.88</td>
<td>20.69</td>
</tr>
<tr>
<td>9</td>
<td>1.45</td>
<td>1.51</td>
<td>20.39</td>
</tr>
<tr>
<td>10</td>
<td>1.46</td>
<td>1.49</td>
<td>20.39</td>
</tr>
<tr>
<td>11</td>
<td>1.47</td>
<td>1.47</td>
<td>20.39</td>
</tr>
<tr>
<td>12</td>
<td>1.48</td>
<td>1.45</td>
<td>20.39</td>
</tr>
</tbody>
</table>

The optimum appears to be 20.39 ft$^2$, and it is found at a value of $r$ approximately equal to 1.45 or 1.46 ft. To get more accuracy, we could refine the search.
Example 5.10 — Search for Two Variables

Problem:
Find the values of x and z (both > 0) that maximize

\[ U = -x^2 + 10x + xz - z^2 + 8z + 2 \]

Solution:

Preparation:
- Re-read it and look at it closely:

Find the values of x and z (both > 0) that maximize

\[ U = -x^2 + 10x + xz - z^2 + 8z + 2 \]

Note the two design variables x and z. Note, too, that there are no constraints that can be used to eliminate one of them.

- Rewrite the given info

\[ U = -x^2 + 10x + xz - z^2 + 8z + 2 \]

- To Find: Values of x and z that maximize U. (x and z are both >0.)

Translation to an optimization problem:

Formulation as an optimization problem:

Criterion function:
\[ U = -x^2 + 10x + xz - z^2 + 8z + 2 \]

Design variables: x, z

Constraints: x, z positive

Application:

We will solve this using calculus.

(To solve the problem graphically requires three dimensions (x, z, U). There are methods for doing this but they are beyond our interest.)

\[ \frac{\partial U}{\partial x} = (-2x + 10 + z) = 0 \]

\[ \frac{\partial U}{\partial z} = x - 2z + 8 = 0 \]

Solving the above two equations for x and z gives:

\[ x = 9.333 \quad \text{and} \quad z = 8.667 \]

With these values, \( U = 83.33 \).

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